Centre Number

Student Number

SCEGGS Darlinghurst

2009

HSC Assessment 2

26th June 2009

Extension 1 Mathematics

Assessment Outcomes: HE4, HE6, HE7

Weighting: 30%

General Instructions

- Time allowed – 70 Minutes
- This paper has **four** questions
- Answer on the paper provided
- Start each question on a **new page**
- Write your Student Number at the top of each page
- Attempt **all** questions and show all necessary working
- Marks may be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used
- A table of standard integrals is provided at the back of this paper

Total marks (46)

- Attempt Questions 1 – 4

Question	Communication	Calculus	Reasoning	Marks
1	/1	/2	/5	/10
2	/4	/6	/2	/12
3	/1	/7	/3	/11
4	/1	/5	/3	/13
TOTAL	/7	/20	/13	**/46**

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46 marks
Attempt Questions 1 – 4

Answer each question on the paper provided
Write your Student Number at the top of each page
Start each question on a new page

Marks

Question 1 (10 Marks)

a) Find: $\displaystyle\int \frac{dx}{\sqrt{9-16x^2}}$ **2**

b) Consider the function $f(x)=e^x-3x$

i) Show a root of $f(x)=0$ lies between $x=0$ and $x=1$ **1**

ii) Apply the 'halving the interval' method twice to find a smaller interval within which the root lies **2**

c) Find the exact value:

$$\sin\left(2\cos^{-1}\left(-\frac{5}{6}\right)\right)$$ **2**

d) Find the term independent of x in the expansion of $\left(2x^3-\frac{1}{x}\right)^{12}$ **3**

Start A New Page **Marks**

Question 2 (12 Marks)

a) Using the substitution $x = \cos\theta$, evaluate $\int_0^{1/2} 2\sqrt{1-x^2}\,dx$ **4**

b) i) State the domain and range of the function $y = 2\sin^{-1}\left(\frac{x}{2}-1\right)$ **2**

ii) Sketch the graph of $y = 2\sin^{-1}\left(\frac{x}{2}-1\right)$ clearly labelling all intercepts. **1**

c) i) Use one application of Newton's method to find an approximation of the root of the equation $x^3 - 12x + 2 = 0$ near $x = 3.1$. (to 1 decimal place) **2**

ii) Sketch the curve of $y = x^3 - 12x + 2$ labelling the turning points. **2**

(Do Not Find The x – Intercepts)

iii) If you choose $x = 1.9$ as your first approximation, explain why you would not arrive at the root near $x = 3.1$ after one application of Newton's Method. **1**

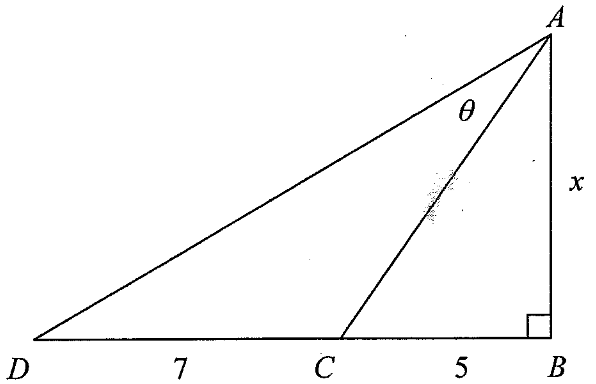
Start A New Page **Marks**

Question 3 (11 Marks)

a) Use the substitution $x = u^2$ to find $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$ **3**

b) Find the greatest coefficient of $(4x^2 + 3x)^9$ **3**

c) In the diagram, DCB is a straight line, $\angle ABC = 90^o$, $\angle CAD = \theta$
$AB = x$, $DC = 7$ and $CB = 5$



i) Show that $\theta = \tan^{-1}\left(\dfrac{12}{x}\right) - \tan^{-1}\left(\dfrac{5}{x}\right)$ **1**

ii) Show that $\dfrac{d\theta}{dx} = \dfrac{5}{x^2+25} - \dfrac{12}{x^2+144}$ **2**

iii) Find the value of x which maximises θ **2**

Start A New Page **Marks**

Question 4 (13 Marks)

a) Consider the binomial expansion: **3**

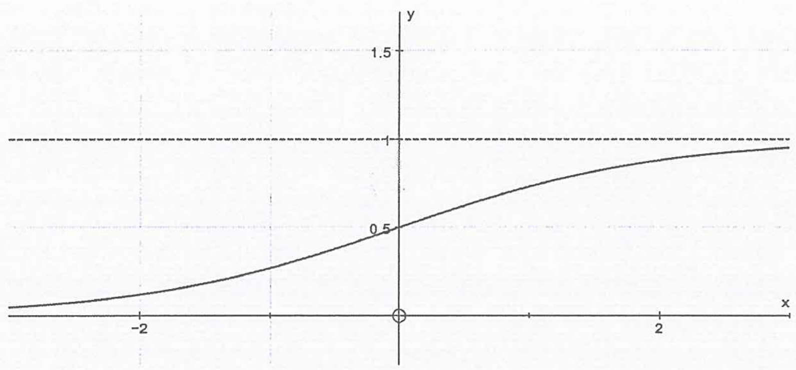
$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots\ldots+\binom{n}{n}x^n$$

Show that $1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots\ldots-(-1)^{n+1}\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$

b) Consider the function $f(x)=\dfrac{e^x}{1+e^x}$

i) Show that $y=f(x)$ is monotonically increasing and hence explain why $f^{-1}(x)$ exists. **2**

The graph of $f(x)$ is below



ii) Sketch $y=f^{-1}(x)$ **1**

iii) Show algebraically that $f^{-1}(x)=\ln\left(\dfrac{x}{1-x}\right)$ **2**

iv) Find $f^{-1}\left(\dfrac{1}{2}\right)$ and $f^{-1}\left(\dfrac{5}{6}\right)$ **2**

v) Evaluate $\displaystyle\int_{1/2}^{5/6} f^{-1}(x)\,dx$ **3**

End Of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension 1 Mathematics Assessment Task 2 2009 – Solutions

Q1 a) $\displaystyle\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{9/16 - x^2}}$ ✓

Calc – 2

$\displaystyle = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + c$ ✓

– those who manipulated the integral algebraically did better than those who remembered the form.

b) i) $f(0) = e^0 - 3\times 0 = 1 > 0$ $\quad f(1) = e^1 - 3\times 1 \doteqdot -0.28 < 0$

since $f(0) > 0$ and $f(1) < 0$ and $y = f(x)$ is continuous a root lies between $x = 0$ and $x = 1$ ✓

Comm – 1

– must mention that $f(x)$ is continuous.

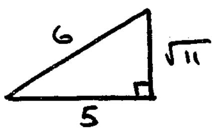
ii)

x	0	1	$\frac{1}{2}$	$\frac{3}{4}$
$f(x)$	1	-0.28	0.149	-0.133

✓✓

$\therefore$ root lies between 0.5 and 0.75.

c) let $\alpha = \cos^{-1}\left(-\frac{5}{6}\right)$ $\quad \frac{\pi}{2} < \alpha < \pi$



$\therefore \sin\alpha = \frac{\sqrt{11}}{6}$ ✓

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= \cancel{2}\times\frac{\sqrt{11}}{\cancel{6}\,3}\times -\frac{5}{6}$

Reas – 2

$= -\frac{5\sqrt{11}}{18}$ ✓

– done well just be careful of silly mistakes.

d) $\displaystyle\left(2x^3 - \frac{1}{x}\right)^{12} = \sum_{r=0}^{12}\binom{12}{r}(2x^3)^{12-r}\left(-\frac{1}{x}\right)^r$

find r : $(2x^3)^{12-r}\left(-\frac{1}{x}\right)^r = x^0$

$36 - 3r - r = 0$ ✓

$36 - 4r = 0$

$4r = 36$

$r = 9$ ✓

$\therefore$ the term is $\binom{12}{9}(2x^3)^3\left(-\frac{1}{x}\right)^9$

$= -1760$ ✓ Reas – 3

DON'T FORGET THE NEGATIVE SIGN.

Q2 a) $\displaystyle\int_0^{1/2} 2\sqrt{1-x^2}\,dx$ let $x = \cos\theta$ $\quad \frac{1}{2} = \cos\theta \quad 0 = \cos\theta$

$dx = -\sin\theta\, d\theta \quad \theta = \frac{\pi}{3} \quad \theta = \frac{\pi}{2}$

$= \displaystyle\int_{\pi/2}^{\pi/3} -2\sin\theta\sqrt{1-\cos^2\theta}\,d\theta$ ✓

$= \displaystyle\int_{\pi/3}^{\pi/2} 2\sin^2\theta\,d\theta$

$= \displaystyle\int_{\pi/3}^{\pi/2} 1-\cos 2\theta\,d\theta$ ✓

$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$ ✓

$= \left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right)$

$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}$ ✓ Calc – 4

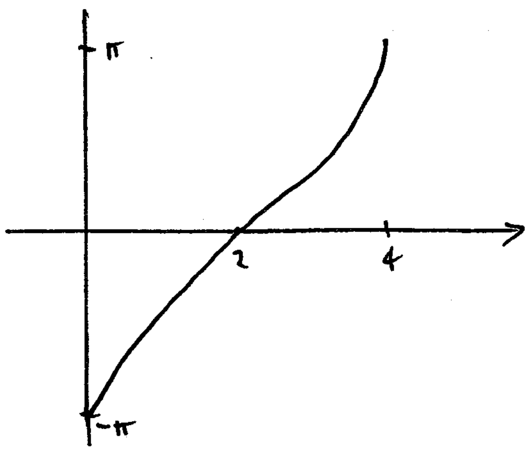
- With the substitution $x = \cos\theta$, when $x = 0$ $\theta = \pi/2$ NOT 1!!!
- Many forgot to substitute dx
- The majority of students could not correctly substitute and ended up with a much easier integral – and this unfortunately could not get you any marks. You needed to integrate $\sin^2\theta$.

b) i) D: $-1 \le \frac{x}{2} - 1 \le 1$ R: $-\pi \le y \le \pi$ ✓

$0 \le \frac{x}{2} \le 2$

$0 \le x \le 4$ ✓ Reas – 2

ii)



✓ Comm – 1

Be careful to draw the $\cos^{-1}x$ shape (& not the $\cos x$ shape

c) i) let $f(x) = x^3 - 12x + 2$

$f'(x) = 3x^2 - 12$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_1 = 3.1 - \dfrac{[(3.1)^3 - 12(3.1) + 2]}{3(3.1)^2 - 12}$ ✓

$= 3.1 - \left(\dfrac{-5.409}{16.83}\right)$

Calc-2

$= 3.4$ ✓

Almost 100% of students got this correct.

ii) turning pts: $\frac{dy}{dx} = 0$

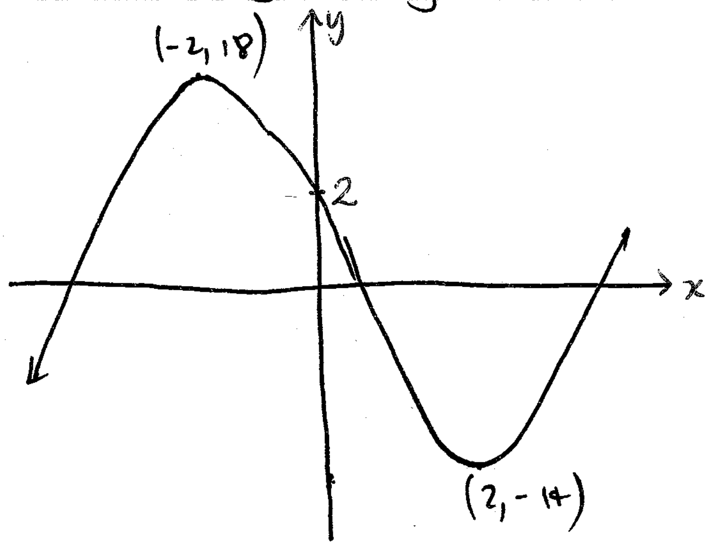
$3x^2 - 12 = 0$

$3(x-2)(x+2) = 0$

$x = 2 \quad x = -2$

$y = -14 \quad y = 18$ ✓

Mostly done well ... but I'm ~~surprised how many~~ did not even calculate the y intercept.



Comm-2

✓

iii) As it is on the other side of the turning point the tangent would get the intercept near another root. ✓ Comm-1

Just because $x = 1.9$ is near the turning point doesn't mean anything, the point is it's on the wrong side. Also, the tangent while almost horizontal, will still cut the x axis

Q3 a) $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$ $\quad x = u^2$, $\quad dx = 2u\,du$

$= \displaystyle\int \frac{2u\,du}{u(1+u^2)}$ ✓

$= \displaystyle\int \frac{2du}{1+u^2}$

$= 2\tan^{-1}u + c$ ✓

Calc-3

$= 2\tan^{-1}\sqrt{x} + c$ ✓

- Students need to recognise integrals. So many didn't see this was an inverse tan solution.

b) $(4x^3+3x)^9 \quad T_{k+1} = {}^9C_k(4x^3)^{9-k}(3x)^k$

$C_{k+1} = {}^9C_k(4)^{9-k}(3)^k \qquad C_k = {}^9C_{k-1}4^{9-(k-1)}3^{k-1}$

$$\frac{C_{k+1}}{C_k} = \frac{\frac{9!}{(9-k)!\,k!} \times 4^{9-k}3^k}{\frac{9!}{[9-(k-1)]!\,(k-1)!} 4^{9-(k-1)}3^{k-1}}$$

$$= \frac{10-k}{k}\cdot\frac{3}{4}$$

$$= \frac{30-3k}{4k} \quad ✓$$

$\therefore \frac{C_{k+1}}{C_k} > 1 \quad 30-3k > 4k$

$7k < 30$

$k < 4\frac{2}{7}$ ✓

$k = 4$

Coefficient $= {}^9C_4 \times 4^5 \times 3^4 = 10\,450\,944$ ✓

Reas - 3

Done very well. Just remember to take your time as one algebraic mistake can be costly.

c) i) $\tan\angle CAB = \frac{5}{x} \qquad \tan\angle DAB = \frac{12}{x}$

$\angle CAB = \tan^{-1}\left(\frac{5}{x}\right) \qquad \angle DAB = \tan^{-1}\left(\frac{12}{x}\right)$ ✓

$\theta = \angle DAB - \angle CAB$

$= \tan^{-1}\left(\frac{12}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right)$ Comm-1

- students thought this was harder than it really was. Read the question carefully

ii) $\theta = \tan^{-1}\left(\frac{12}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right)$

$$\frac{d\theta}{dx} = \frac{-12x^{-2}}{1+\left(\frac{12}{x}\right)^2} - \frac{-5x^{-2}}{1+\left(\frac{5}{x}\right)^2} \quad ✓$$

$$= \frac{-12/x^2}{1+\frac{144}{x^2}} + \frac{5/x^2}{1+25/x^2}$$

Calc - 2

$$= \frac{-12/x^2}{\frac{x^2+144}{x^2}} + \frac{5/x^2}{\frac{x^2+25}{x^2}} \quad ✓$$

$$= \frac{5}{x^2+25} - \frac{12}{x^2+144}$$

- In a show question you need to give a more detailed solution

iii) find max $\frac{dx}{d\theta}=0$

$$\frac{5}{x^2+25}-\frac{12}{x^2+144}=0$$

$$\frac{5}{x^2+25}=\frac{12}{x^2+144}$$

$$5x^2+720=12x^2+300$$

$$7x^2=420$$

$$x^2=60$$

$$x=\pm\sqrt{60}$$ ✓ $x=\sqrt{60}$ as x represents length.

This part was done well if you got up to it.

test $x=\sqrt{60}$

x	7	$\sqrt{60}$	8
$\frac{d\theta}{dx}$	0.0053	0	−0.0015

✓

$\therefore x=\sqrt{60}$ gives a maximum. Calc - 2

Q4 a) $$\int(1+x)^n=\int 1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n$$

$$\frac{(1+x)^{n+1}}{n+1}+c=x+\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}+\dots+\binom{n}{n}\frac{x^{n+1}}{n+1}$$ ✓

This question was very well done.

let $x=0$ to find c

$$c=\frac{-1}{n+1}$$ ✓

$$\therefore\frac{(1+x)^{n+1}}{n+1}-\frac{1}{n+1}=x+\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}+\dots+\binom{n}{n}\frac{x^{n+1}}{n+1}$$

let $x=-1$

$$\frac{-1}{n+1}=-1+\frac{1}{2}\binom{n}{1}-\frac{1}{3}\binom{n}{2}+\dots+\frac{(-1)^{n+1}}{n+1}\binom{n}{n}$$ ✓

$$1-\frac{1}{2}\binom{n}{1}-\frac{1}{3}\binom{n}{2}+\dots-(-1)^{n+1}\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$$

Reas - 3

b) i) $y = \frac{e^x}{e^x+1}$

$\frac{dy}{dx} = \frac{(e^x+1)e^x - e^x \times e^x}{(e^x+1)^2}$

$= \frac{e^{2x}+e^x - e^{2x}}{(e^x+1)^2}$

$= \frac{e^x}{(e^x+1)^2}$ ✓

since $e^x > 0$ for all x

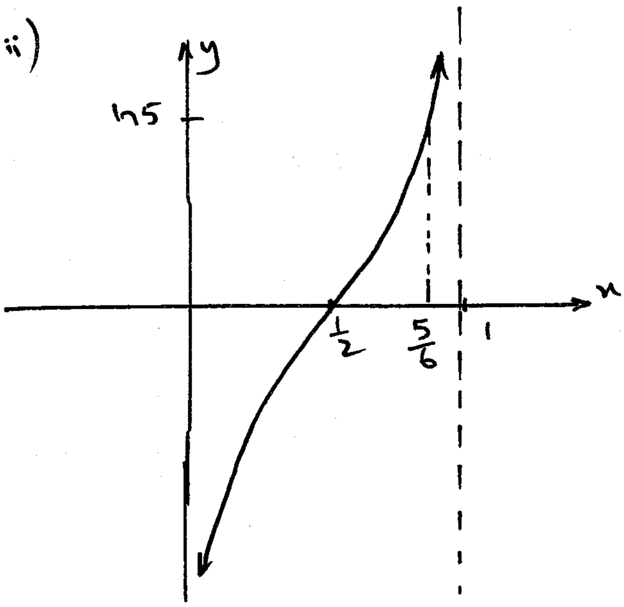
$\frac{dy}{dx} > 0$ for all x

$\therefore$ the curve of $y=f(x)$ is monotonically increasing and hence has an inverse function as it satisfies the horizontal line test. ✓

Calc-2

Note: every x has a unique y is NOT the horizontal line test (it's the vertical line test)

ii)



✓

Comm-1

There were some very sloppy graphs. Use a pencil & a ruler, label the x intercept and label the asymptote

iii) $y = \frac{e^x}{1+e^x}$

interchange x and y

$x = \frac{e^y}{1+e^y}$ ✓

$x + xe^y = e^y$

$x = e^y - xe^y$

$= e^y(1-x)$ ✓

There was a lot of fudging for that 2nd mark!

$e^y = \frac{x}{1-x}$

$y = \ln\left(\frac{x}{1-x}\right)$

iv) $f^{-1}\left(\frac{1}{2}\right) = \ln\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)$

$= \ln 1$

$= 0$ ✓

$f^{-1}\left(\frac{5}{6}\right) = \ln\left(\frac{5/6}{1-5/6}\right)$

$= \ln\left(\frac{5/6}{1/6}\right)$

$= \ln 5$ ✓

v) $\int_{\frac{1}{2}}^{5/6} f^{-1}(x)\,dx = \underbrace{\frac{5}{6}\ln 5}_{\text{Area } \square} - \int_0^{\ln 5} \frac{e^x}{1+e^x}\,dx$ ✓

$= \frac{5}{6}\ln 5 - \left[\ln(1+e^x)\right]_0^{\ln 5}$ ✓

$= \frac{5}{6}\ln 5 - \left[\ln(1+e^{\ln 5}) - \ln(1+e^0)\right]$

$= \frac{5}{6}\ln 5 - (\ln 6 - \ln 2)$

$= \frac{5}{6}\ln 5 - \ln 3.$ ✓

Calc - 3

These were 2 easy marks picked up by most in a rush at the end. Well done.

lots of careless mistakes in the last substitution which was a pity.